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Approximation of the semigroup generated by the Hamiltonian of Reggeon field theory in Bargmann space

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Abstract

The Reggeon field theory is governed by a non-self adjoint operator constructed as a polynomial in A, A^*, the standard Bose annihilation and creation operators. In zero transverse dimension, this Hamiltonian acting in Bargmann space is defined by

$$H_{\lambda',\mu} = \lambda' A^{*2} A^2 + \mu A^* A + i\lambda A^*\big(A^* + A\big)A,$$

where $i^2 = -1$, λ', μ and λ are real numbers and the operators A, A^* satisfy the commutation relation $[A, A^*] = I$. As the quantum mechanical system described by $H_{\lambda',\mu}$ has a velocity-dependent potential containing powers of momentum up to the fourth, the problem of existence of Hamiltonian path integral for the evolution operator $e^{-tH_{\lambda',\mu}}$ of this theory is of interest on its own. In particular, can we express $e^{-tH_{\lambda',\mu}}$ as a limit of "integral" operators? In this article one considerably reduces the difficulty by studying the Trotter product formula of $H_{\lambda',\mu}$ to reach two objectives:

- The first objective is to prove a very specific error estimate for the error in a Trotter product formula in trace-norm for H viewed as the sum of the operators $\lambda' A^{*2} A^2$ and $\mu A^* A + i\lambda A^* \times (A^* + A)A$.
- The second objective of this work is to give a approximation of the semigroup generated by $H_{\lambda',\mu}$ when $H_{\lambda',\mu}$ is split in the sum of $\lambda' A^{*2} A^2 + \mu A^* A$ and $i\lambda A^*(A^* + A)A$. We note

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that this case is entirely different. In fact, the usual Trotter product formula is not defined, because the interaction operator $A^*(A^*+A)A$ is not the infinitesimal generator of a semigroup on Bargmann space. For $\lambda' > 0$ and $\varepsilon > 0$, we choose an approximation operator $\theta_\varepsilon = [I - \varepsilon i\lambda A^* \times (A^*+A)A]e^{-\varepsilon(\lambda' A^{*2}A^2+\mu A^*A)}$ and we give a connection between θ_ε and $e^{-\varepsilon H_{\lambda',\mu}}$. This choice allows us to give in [A. Intissar, Note on the path integral formulation of Reggeon field theory, preprint] a "generalized Trotter product formula" for $T_\mu = \mu A^*A + i\lambda A^*(A+A^*)A$, i.e., for limit case as $\lambda' = 0$ and answers to the above question.

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1. Introduction

Reggeon field theory is one of the attempts to understand strong interactions, i.e., the interaction between, among other less stable particles, protons and neutrons. This theory (in zero transverse dimension) is governed by a non-self adjoint Gribov operator

$$H_{\lambda',\mu} = \lambda' A^{*2}A^2 + \mu A^*A + i\lambda A^*\big(A^*+A\big)A$$

constructed as a polynomial in A, A^*, the standard Bose annihilation and creation operators acting in Bargmann space. μ is the Pomeron intercept, λ' is a four Pomeron coupling and $i\lambda$ an imaginary Pomeron coupling. We just refer to Gribov in [18] for the fundamentals of Reggeon field theory and to Intissar–Le Bellac–Zerner in [25] for other references developing this theory.

We define the Bargmann space E by

$$E = \left\{\phi : \mathbb{C} \to \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2}\big|\phi(z)\big|^2\,dz\,d\bar{z} < \infty \text{ and } \phi(0) = 0\right\}.$$

E is a Hilbert space for the scalar product $\langle\cdot,\cdot\rangle$ defined by

$$\langle\phi,\psi\rangle = \int_{\mathbb{C}} e^{-|z|^2}\phi(z)\bar{\psi}(z)\,dz\,d\bar{z}.$$

The associated norm is denoted by $\|\cdot\|$. An orthonormal basis of E is given by

$$e_n(z) = \frac{z^n}{\sqrt{n!}}, \quad n = 1, 2, \ldots.$$

The space E was used in [7] as representation space for the canonical commutation rules of quantum mechanics. Since then, it had occurred in many different contents, i.e., in representation theory of nilpotent Lie groups and as a class of symbols in the theory of Hankel and Toeplitz operators.

In this representation, the annihilation and creation operators are defined by

$$A\phi(z) = \phi'(z) \quad \text{with domain } D(A) = \{\phi \in E;\ \phi' \in E\},$$

$$A^*\psi(z) = z\psi(z) \quad \text{with domain } D\big(A^*\big) = \{\psi \in E;\ z\psi \in E\}.$$

Therefore, the Gribov operator $H_{\lambda',\mu}$ can be written as

$$H_{\lambda',\mu}\phi(z) = \big(\lambda' z^2 + i\lambda z\big)\phi''(z) + \big(i\lambda z^2 + \mu z\big)\phi'(z),$$

where ϕ' and ϕ'' are the first and second derivatives of $\phi(z)$ at z.

We denote by $D_{\max}$ the maximal domain of $H_{\lambda',\mu}$ and by $D_{\min}$ its minimal domain; they are defined by

$$D_{\max} = D_{\max}(H_{\lambda',\mu}) = \{\phi \in E \text{ such that } H_{\lambda',\mu}\phi \in E\},$$

$$D_{\min} = D_{\min}(H_{\lambda',\mu}) = \Big\{\phi \in E \text{ such that there exist } p_n \in P_0 \text{ and } \psi \in E \text{ with } p_n \xrightarrow[n\to\infty]{} \phi \text{ and } H_{\lambda',\mu} p_n \xrightarrow[n\to\infty]{} \psi\Big\}$$

where P_0 is the space of polynomials vanishing at the origin.

Remark 1.1.

(1) In Bargmann space, the adjoint of the derivation operator $A = \frac{d}{dz}$ is the multiplication operator $A^* = z$.
(2) The injection from $D(A)$ into the Bargmann space E is compact.
(3) $\|\phi\| \leqslant \|A\phi\|$ for all $\phi \in D(A)$ and $\|\phi\| \leqslant \frac{1}{\sqrt{2}}\|A^*\phi\|$ for all $\phi \in D(A^*)$.
(4) $A^{*2}A^2$ and A^*A are selfadjoint operators with compact resolvent.

In the following, we adopt the notations:

– Let T be a closed linear operator with dense domain $D(T)$ in a Hilbert space H. We denote by $\rho(T)$, $\sigma(T)$, $\mathrm{Im}(T)$ and $\mathrm{Ker}(T)$, the resolvent set, spectrum, range and null space of T, respectively.
– For $\lambda' \geqslant 0$, $\mu > 0$ and $\lambda \in \mathbb{R}$, we write $H_{\lambda',\mu}$ in the following form: $H_{\lambda',\mu} = \lambda' S + \mu U + i\lambda H_I$ where $S = A^{*2}A^2$ with domain $D(S) = \{\phi \in E;\ S\phi \in E\}$ is the quartic part of $H_{\lambda',\mu}$, $U = A^*A$ with domain $D(U) = \{\phi \in E;\ U\phi \in E\}$ is the quadratic part of $H_{\lambda',\mu}$ and $H_I = A^*(A^* + A)A$ with domain $D(H_I) = \{\phi \in E$ such that $H_I\phi \in E\}$ is the cubic part of $H_{\lambda',\mu}$.
– For $\lambda' > 0$ and $\mu > 0$, we set $G = \lambda' S + \mu U$, $T_\mu = \mu U + i\lambda H_I$ and $H_{\lambda'} = \lambda' S + T_\mu$ so that we can write also $H_{\lambda'}$ as $H_{\lambda'} = G + i\lambda H_I$.

Ando–Zerner in [6] and Intissar in [21–24] have given a series of spectral properties for the operator $H_{\lambda'}$, such as definition of the space of states in the Hamiltonian formalism and formulation of the related eigenvalue problem with a rigorous study of the domain of the Gribov operator $H_{\lambda'}$ and a construction of the associated semigroup as well as asymptotic behaviour of this semigroup.

Now, we begin by recalling some spectral properties of the operators that we have defined above.

Theorem 1.2.

(1) *For $\lambda' \geqslant 0$ and $\mu > 0$, $H_{\lambda',\mu}$ is a maximal accretive operator.*

(2) $]-\infty, 0] \subset \rho(H_{\lambda',\mu})$.
(3) $D_{\max}(H_{\lambda'}) = D_{\min}(H_{\lambda'}) = D(S)$.
(4) $D_{\max}(T_\mu) = D_{\min}(T_\mu)$.
(5) $-H_{\lambda',\mu}$ *generates a contraction compact semigroup* $e^{-tH_{\lambda',\mu}}$.
(6) *For each* $\varepsilon > 0$ *there is a constant* $C_\varepsilon > 0$ *such that*

$$\|T_\mu\phi\| \leqslant \varepsilon\|S\phi\| + C_\varepsilon\|\phi\| \quad \text{for every } \phi \in D(S).$$

(7) $-H_{\lambda'}$ *generates an analytic semigroup* $e^{-tH_{\lambda'}}$.
(8) $H_{\lambda'}^{-1}$ *belongs to Carleman-class* C_p *for all* $p > \frac{1}{2}$ *and* $T_\mu^{-1} \in C_{1+\varepsilon}$ *for every* $\varepsilon > 0$, *where* C_p *denotes the space of all compact operators* K *such that* $\sum_{n=1}^{\infty} s_n^p(K) < \infty$ *and* $s_n(K)$ *are known as the eigenvalues of the compact selfadjoint operator* $\sqrt{K^*K}$ *or as the singular values of the operator* K.
(9) *The system of root vectors of* $H_{\lambda'}$ *is dense in Bargmann space* E.
(10) *The spectrum of* T_μ *is real and the spectral radius of* $H_{\lambda'}^{-1}$ *converges to the spectral radius of* T_μ^{-1} *when* λ' *goes to zero.*

Remark 1.3.

(i) The spectral properties (1) to (6) of the above theorem are established by Intissar in [21] and the property (7) is shown in [22, Theorem 1].
(ii) The spectral properties (8) and (9) are established by Aimar–Intissar–Paoli in [1,3] (see also [2,4,5]).
(iii) The spectral properties in (10) are shown in [6] and [22].
(iv) For a systematic treatment of operators of Carleman-class, we refer to the Gohberg and Krein's book [17] or to the Dunford and Schwartz's book [15]. See also [23,27] for concrete applications.

We now describe briefly the contents of this paper, section by section.

In Section 2, we give some approximation results which estimate the convergence of $H_{\lambda'}^{-1}$ to T_μ^{-1} with respect to the parameter λ'. This result is essentially based on the inclusion of $D(T_\mu)$ in $D(S^{1/2})$ (see Lemma 2.1).

If $H_{\lambda'}$ is split in the sum of $\lambda' A^{*2}A^2$ and $\mu A^*A + i\lambda A^*(A^* + A)A$, i.e., $H_{\lambda'} = \lambda' S + T_\mu$, we show that the convergence of the usual Trotter product formula for $H_{\lambda'}$ is of classical type and can be lifted to trace-norm convergence. We also give some error bounds in this last topology. This result is deduced by applying Theorem 3 in [8]. In particular, we show there exist $t_0 > 0$ such that the Trotter formula for $H_{\lambda'} = \lambda' S + T_\mu$ (in despite of T_μ is not selfadjoint) converges in trace-norm uniformly for $t \geqslant t_0$ and $C > 0$ such that

$$\left\|\left(e^{-(t/n)\lambda' S} e^{-(t/n)T_\mu}\right)^n - e^{-tH_{\lambda'}}\right\|_1 \leqslant C\frac{\log n}{n}, \quad n = 2, 3, \ldots \text{ for } t \geqslant t_0.$$

In Section 3, we consider $H_{\lambda'}$ as the sum of $\lambda' A^{*2}A^2 + \mu A^*A$ and $i\lambda A^*(A^* + A)A$, i.e., $H_{\lambda'} = G + i\lambda H_I$; the results of Cachia–Neidhardt–Zagrebnov [8–10,30–33] are not applicable, because H_I is chaotic, see [13,24], and is not the infinitesimal generator of a semigroup on Bargmann space. This remark is an argument among others for which the usual Trotter product formula does not hold.

However, the usual procedure to define a path integral for the Hamiltonian $H_{\lambda'}$ is to look for an operator θ_ε such that

$$\operatorname{Lim}(\theta_\varepsilon)^n = e^{-tH_{\lambda'}}, \quad \varepsilon = \frac{t}{n} \text{ as } n \to \infty \text{ in strong or weak sense.} \tag{$*$}$$

As

$$H_{\lambda'} = \left(\lambda' z^2 + i\lambda z\right)\left(\frac{d}{dz}\right)^2 + (\mu z + i\lambda z)\frac{d}{dz}$$

whose matrix element between coherent states is

$$\frac{\langle z|H_\lambda'|\xi\rangle}{\langle z|\xi\rangle} = \left(\lambda' z^2 + i\lambda z\right)\bar{\xi}^2 + (\mu z + i\lambda z)\bar{\xi} = h(z,\xi).$$

Here the coherent state vector $|\xi\rangle$ is defined by

$$|\xi\rangle = e^{-(1/2)|\xi|^2} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |n\rangle,$$

where $\{|n\rangle;\ n = 0, 1, 2, \ldots\}$ are the eigenvectors of the positive semi-definite number operator A^*A and

$$\langle z|\xi\rangle = e^{-(1/2)|z-\xi|^2 + i\,\mathrm{Im}(\bar{z}\xi)}$$

and the quantities which have physical interpretation for high-energy scattering process are $\langle 0|e^{i\alpha A} e^{-tH_{\lambda'}} e^{-i\beta A^*}|0\rangle$ where $(\alpha, \beta) \in \mathbb{C}^2$ and $|0\rangle$ is the vacuum state defined by $A|0\rangle = 0$.

By inserting the coherent states completeness relation in $(*)$, we get

$$\left\langle 0\middle| e^{-i\alpha A} e^{-tH_\lambda'} e^{-i\beta A^*} \middle|0\right\rangle = \lim_{n\to\infty} \int_{\mathbb{C}} \ldots \int_{\mathbb{C}} \prod_{k=1}^{n} \frac{dz_k d\bar{z}_k}{2i\pi} N_\varepsilon(z_{k+1}, \bar{z}_k) e^{(z_{k+1}-z_k)\bar{z}_{k_e} - i\beta z_1},$$

where

$$N_\epsilon(z, \bar{\xi}) = \frac{\langle z|\theta_\varepsilon|\xi\rangle}{\langle z|\xi\rangle} \quad \text{and} \quad z_{n+1} = -i\alpha.$$

The condition $\theta_\varepsilon = I - \varepsilon H + o(\varepsilon)$ imposed by $(*)$ will not insure in general the existence of the limit. A natural choose of $N_\varepsilon(z, \bar{\xi})$ is $e^{-\varepsilon h(z,\bar{\xi})}$ but this kernel is not in Bargmann space for sufficiently large $\bar{\xi}$.

In the following, we propose to use the exact evolution kernel for G in order to regularise the interaction term H_I and choose an approximation operator as follows:

$$\theta_\varepsilon = [I - i\varepsilon\lambda H_I] e^{-\varepsilon G}$$

or

$$\theta_\varepsilon = e^{-\varepsilon G} - \varepsilon i\lambda A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G} A\right)A.$$

Then for every $\phi \in D(G^3)$, we can give an estimation of the difference $\theta_\varepsilon\phi - e^{-\varepsilon H_{\lambda'}}\phi$ to order 2 with respect to the parameter ε.

This estimation allows us in [26] to prove a "generalized Trotter product formula" for $H_{\lambda'}$ with $\lambda' \geqslant 0$, i.e., $\operatorname{Lim}(\theta_\varepsilon)^n = e^{-tH_{\lambda'}}$, $\varepsilon = \frac{t}{n}$ as $n \to \infty$ in strong sense by applying the General Representation Theorem 5.3 in [35].

2. The usual Trotter product formula for the Gribov operator

We begin this section by giving some approximation results which estimate the convergence of $H_{\lambda'}^{-1}$ to T_μ^{-1} with respect to the parameter λ'. We start with lemma.

Lemma 2.1. *Let $S = A^{*2}A^2$ acting on Bargmann space E with domain $D(S) = \{\phi \in E;\ S\phi \in E\}$.*

Then

(1) $D(T_\mu) \subset D(S^{1/2})$ *for* $\mu \neq 0$.
(2) $D(S^\alpha) \subset D(T_\mu)$ *for all* $\alpha \geqslant \frac{3}{4}$ *and* $\mu \neq 0$.

Proof. (1) Let $H_I^{\min} = A^*(A^* + A)A$ with minimal domain $D_{\min}(H_I) = \{\phi \in E$ such that there exist $p_n \in P_0$ and $\psi \in E$ with $p_n \xrightarrow[n\to\infty]{} \phi$, and $H_I p_n \xrightarrow[n\to\infty]{} \psi\}$ where P_0 is the space of polynomials vanishing at the origin, then the domain of T_μ is $D(T_\mu) = D(A^*A) \cap D(H_I^{\min})$ (see Zerner's lemma in [2]) and as $D(A^*A) = D(S^{1/2})$ we deduce that $D(T_\mu) \subset D(S^{1/2})$.

(2) As an orthonormal basis of E is given by $e_n(z) = \frac{z^n}{\sqrt{n!}}$, $n = 1, 2, \ldots$, then for $\phi \in E$, we have

$$\phi(z) = \sum_{k=1}^{\infty} a_k e_k(z)$$

and

$$\phi \in D(T_\mu) \quad \text{if and only if}$$
$$\sum_{k=1}^{\infty} \bigl|\mu k a_k + i\lambda\bigl[(k-1)\sqrt{k}a_{k-1} + k\sqrt{k+1}a_{k+1}\bigr]\bigr|^2 < \infty.$$

In particular,

$$\left\{\phi \in E;\ \sum_{k=1}^{\infty} \bigl|k^{3/2}a_k\bigr|^2 < \infty\right\} \subset D(T_\mu).$$

Now, as $Se_k(z) = k(k-1)e_k(z)$, it follows that

$$\phi \in D\bigl(S^\alpha\bigr) \quad \text{if and only if} \quad \sum_{k=1}^{\infty} k^{4\alpha}|a_k|^2 < \infty.$$

So if $4\alpha \geqslant 3$, there exist $C_{\mu\lambda} > 0$ and $M_{\mu\lambda} > 0$ such that

$$\sum_{k=1}^{\infty} \bigl|\mu k a_k + i\lambda\bigl[(k-1)\sqrt{k}a_{k-1} + k\sqrt{k+1}a_{k+1}\bigr]\bigr|^2 \leqslant C_{\mu\lambda} \sum_{k=1}^{\infty} k^3|a_k|^2$$
$$\leqslant M_{\mu\lambda} \sum_{k=1}^{\infty} k^{4\alpha}|a_k|^2. \qquad \square$$

Remark 2.2.

(i) If we write $\xi = \operatorname{Re}\xi + i\operatorname{Im}\xi$ with $\operatorname{Re}\xi > 0$, we get

$$\|T_\mu\phi + \xi\phi\|^2 = \|T_\mu\phi + i\operatorname{Im}\xi\phi\|^2 + 2\operatorname{Re}\xi\operatorname{Re}\langle T_\mu\phi, \phi\rangle + (\operatorname{Re}\xi)^2\|\phi\|^2,$$

and as $\operatorname{Re}\langle T_\mu\phi, \phi\rangle \geqslant 0$, we have

$$\|T_\mu\phi + i\operatorname{Im}\xi\phi\| \leqslant \|T_\mu\phi + \xi\phi\| \quad \text{for all } \phi \in D(T_\mu).$$

(ii) Always by virtue of $D(T_\mu) \subset D(S^{1/2})$ we deduce there is a constant $C > 0$ such that

$$\|S^{1/2}\phi\| \leqslant C\big(\|T_\mu\phi\| + \|\phi\|\big) \quad \text{for all } \phi \in D(T_\mu).$$

The above lemma allows us to estimate the convergence of $H_{\lambda'}^{-1}$ to T_μ^{-1} with respect to the parameter λ' and improves the result of Intissar given in [21, Lemma 11]. The estimate of the convergence of $H_{\lambda'}^{-1}$ to T_μ^{-1} with respect λ' is the contents of our next theorem.

Theorem 2.3. *Let $\lambda' > 0$ and $\mu > 0$. Then for every ξ with $\operatorname{Re}\xi > 0$, there is a constant $C_\xi > 0$ such that $\|(H_{\lambda'} + \xi I)^{-1} - (T_\mu + \xi I)^{-1}\| \leqslant C_\xi\sqrt{\lambda'}$.*

Proof. Let $\xi \in \mathbb{C}$ with $\operatorname{Re}\xi > 0$. Then for all $\psi \in E$, there exist $\phi \in D(T_\mu)$ and $\phi_{\lambda'} \in D(S)$ such that

$$T_\mu\phi + \xi\phi = \psi \quad \text{and} \quad \lambda' S\phi_{\lambda'} + T_\mu\phi_{\lambda'} + \xi\phi_{\lambda'} = \psi.$$

Hence we can write

$$\xi[\phi - \phi_{\lambda'}] = -T_\mu(\phi - \phi_{\lambda'}) + \lambda' S\phi_{\lambda'}.$$

So we have

$$\xi\|\phi - \phi_{\lambda'}\|^2 = -\big\langle T_\mu(\phi - \phi_{\lambda'}), \phi - \phi_{\lambda'}\big\rangle + \langle\lambda' S\phi_{\lambda'}, \phi - \phi_{\lambda'}\rangle$$

and

$$\operatorname{Re}\xi\|\phi - \phi_{\lambda'}\|^2 = -\operatorname{Re}\big\langle T_\mu(\phi - \phi_{\lambda'}), \phi - \phi_{\lambda'}\big\rangle + \lambda'\operatorname{Re}\langle S\phi_{\lambda'}, \phi - \phi_{\lambda'}\rangle.$$

As $\operatorname{Re}\langle T_\mu(\phi - \phi_{\lambda'}), \phi - \phi_{\lambda'}\rangle \geqslant 0$, we deduce that

$$\operatorname{Re}\xi\|\phi - \phi_{\lambda'}\|^2 \leqslant \lambda'\operatorname{Re}\langle S\phi_{\lambda'}, \phi - \phi_{\lambda'}\rangle.$$

Now, by virtue of $D(T_\mu) \subset D(S^{1/2})$, we obtain

$$\begin{aligned}
\operatorname{Re}\xi\|\phi - \phi_{\lambda'}\|^2 &\leqslant \lambda'\operatorname{Re}\big\langle S^{1/2}\phi_{\lambda'}, S^{1/2}\phi - S^{1/2}\phi_{\lambda'}\big\rangle \\
&\leqslant \lambda'\|S^{1/2}\phi_{\lambda'}\|\,\|S^{1/2}\phi\| - \lambda'\|S^{1/2}\phi_{\lambda'}\|^2 \\
&\leqslant \frac{\lambda'}{2}\|S^{1/2}\phi_{\lambda'}\|^2 + \frac{\lambda'}{2}\|S^{1/2}\phi\|^2 - \lambda'\|S^{1/2}\phi_{\lambda'}\|^2 \\
&\leqslant \frac{\lambda'}{2}\|S^{1/2}\phi\|^2 - \frac{\lambda'}{2}\|S^{1/2}\phi_{\lambda'}\|^2 \\
&\leqslant \frac{\lambda'}{2}\|S^{1/2}\phi\|^2
\end{aligned}$$

and therefore

$$\|\phi - \phi_{\lambda'}\| \leqslant \sqrt{\frac{\lambda'}{2\operatorname{Re}\xi}}\,\big\|S^{1/2}\phi\big\|.$$

So it follows from (i) and (ii) of Remark 2.2 that

$$\begin{aligned}
\big\|S^{1/2}\phi\big\| &\leqslant C\big(\|T_\mu\phi + i\operatorname{Im}\xi\phi - i\operatorname{Im}\xi\phi\| + \|\phi\|\big)\\
&\leqslant C\big(\|T_\mu\phi + i\operatorname{Im}\xi\phi\| + |\operatorname{Im}\xi|\|\phi\| + \|\phi\|\big)\\
&\leqslant C\|T_\mu\phi + \xi\phi\| + C\big(1 + |\operatorname{Im}\xi|\big)\|\phi\|\\
&\leqslant C\|\psi\| + C\big(1 + |\operatorname{Im}\xi|\big)\big\|(T_\mu + \xi I)^{-1}\psi\big\|\\
&\leqslant C\|\psi\| + C\big(1 + |\operatorname{Im}\xi|\big)\frac{1}{\operatorname{Re}\xi}\|\psi\|\\
&\leqslant C\left(1 + \big(1 + |\operatorname{Im}\xi|\big)\frac{1}{\operatorname{Re}\xi}\right)\|\psi\|
\end{aligned}$$

and hence we obtain

$$\|\phi - \phi_{\lambda'}\| \leqslant C\sqrt{\frac{\lambda'}{2\operatorname{Re}\xi}}\left[1 + \big(1 + |\operatorname{Im}\xi|\big)\frac{1}{\operatorname{Re}\xi}\right]\|\psi\|,$$

$$\big\|(T_\mu + \xi I)^{-1} - (H_{\lambda'} + \xi I)^{-1}\big\| \leqslant C_\xi\sqrt{\lambda'} \quad \text{with}$$

$$C_\xi = \frac{C}{\sqrt{2\operatorname{Re}\xi}}\left[1 + \big(1 + |\operatorname{Im}\xi|\big)\frac{1}{\operatorname{Re}\xi}\right]. \qquad \square$$

Corollary 2.4. *$H_{\lambda'}$ converges to T_μ strongly in the generalized sense as $\lambda' \to 0$.*

For the notion of generalized strong convergence of closed linear operators, we refer to Kato's book [28].

Remark 2.5.

(i) With the above operators S and T_μ, we can note that the existence conditions of usual Trotter formula for our Gribov's operator are satisfied, i.e.,

$$\lim\big(e^{-\varepsilon\lambda' A^{*2}A^2} e^{-\varepsilon[\mu A^*A + i\lambda A^*(A+A^*)A]}\big)^n\psi = e^{-tH_{\lambda'}}\psi; \quad t \geqslant 0 \quad \text{and}$$
$$\varepsilon = \frac{t}{n}, \quad n \to \infty.$$

(ii) There exists an extensive literature on sufficient conditions for existence of the usual Trotter–Kato product formulas for the general operators, the reader may be referred to Trotter [39], Kato [29], Reed–Simon [36], Chernoff [11,12], Faris [16] or Ichinose [19].

(iii) In [37], Rogava proved (under some reasonable conditions) that the convergence of the usual Trotter product formula for self-adjoint operators can be lifted to operator-norm convergence. Afterwards, a series of works on this subject have appeared, see for example Neidhardt–Zagrebnov in [30–33], Doumeki–Ichinose–Tamura in [14] and Ichinose-Tamura in [20]. They extended the problem and improved the result of Rogava in various senses. In particular, they established an error estimate for the Trotter product formula in normed ideals.

(iv) As $H_{\lambda'}$ is a non-selfadjoint operator, the results obtained by Rogava and the group of mathematical physicists mentioned above, are not applicable to our operator. Nevertheless, we can apply recent results of Doumeki–Ichinose–Tamura in [14,20] or Cachia–Neidhardt–Zagrebnov in [8–10] to deduce that the convergence of the usual Trotter product formula for our concrete operator can be lifted to trace-norm convergence and we establish, as in the self-adjoint case, an error estimate for the Trotter product formula in trace-norm.

Before checking the assumptions of Theorem 3 in [8] which is recalled in the following, we begin by writing the commutation relations of the operator $S = A^{*2}A^2$ with the operators A, A^*, $U = A^*A$, $H_{12} = A^*A^2$, $H_{21} = A^{*2}A$, $H_I = A^*(A^* + A)A$ and $T_\mu = \mu A^*A + i\lambda A^*(A^* + A)A$.

Lemma 2.6. *Let $S = A^{*2}A^2$, $S_\gamma = \lambda' S + \gamma I$, $\lambda' > 0$, $\gamma > 0$ and $P(A^*, A)$ a polynomial in A and A^* defined by*

$$P\big(A^*, A\big) = aA^* + bA + cA^*A + dA^{*2}A + eA^*A^2,$$

where a, b, c, d and e are real numbers.

Then we have the following properties:

(1) *For all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$\big\| P\big(A^*, A\big)\phi\big\| \leqslant \varepsilon\|S\phi\| + C_\varepsilon\|\phi\| \quad \textit{for every } \phi \in D(S).$$

(2) $SP(A^*, A) = P(A^*, A)S + Q(A^*, A)S + R(A^*, A)$, *where*

$$Q\big(A^*, A\big) = 2\big(dA^* - eA\big)$$

and

$$R\big(A^*, A\big) = 2(a + d)A^{*2}A + 2(e - b)A^*A^2.$$

Proof. (1) It is an immediate consequence of [21, Lemma 6].

(2) By using the commutation relation $[A, A^*] = I$ in Bargmann space, the following relations hold:

(i) $SA = AS - 2A^*A2;$
(ii) $SA^* = A^*S + 2A^{*2}A;$
(iii) $SU = US;$
(iv) $SH_{12} = H_{12}S - 2AS + 2A^*A^2 = H_{12}(S + 2I) - 2AS;$
(v) $SH_{21} = H_{21}S + 2A^*S + 2A^{*2}A = H_{21}(S + 2I) + 2A^*S.$

From the above relations, we get

$$SP\big(A^*, A\big) = P\big(A^*, A\big)S + Q\big(A^*, A\big)S + R\big(A^*, A\big).$$

In particular, we obtain

$$SH_I = H_I S + 2\big(A^* - A\big)S + 2H_I = H_I(S + 2I) + 2\big(A^* - A\big)S \quad \text{and}$$
$$ST_\mu = T_\mu S + 2i\lambda\big[H_I + \big(A^* - A\big)S\big]. \qquad \square$$

Remark 2.7.

(i) The operator S does not commute with the operators A, A^*, H_I and T_μ.
(ii) It follows from the above lemma, that the operators $Q(A^*, A)S_\gamma^{-1}$, $R(A^*, A)S_\gamma^{-1}$ and $P(A^*, A)S_\gamma^{-1}$ are bounded.

Now we recall the result of Cachia–Zagrebnov [8, Theorem 3].

Theorem 2.8. *Let A be a positive self-adjoint operator in a Hilbert space such that its resolvent belongs to C_p for finite $p \geqslant 1$. If B is a maximal-accretive operator such that $D(A) \subset D(B)$ and $D(A) \subset D(B^*)$ with*

$$\|B\varphi\| \leqslant c_1\|A\varphi\|, \quad \varphi \in D(A),\ 0 < c_1 < 1,$$
$$\big\|B^*\varphi\big\| \leqslant c_2\|A\varphi\|, \quad \varphi \in D(A),\ 0 < c_2 < 1,$$

then the Trotter formula converges in trace-norm uniformly for $t \geqslant t_0 > 0$, and we have

$$\big\|\big(e^{-(t/n)A}e^{-(t/n)B}\big)^n - e^{-t(A+B)}\big\|_1 \leqslant O\left(\frac{\log n}{n}\right).$$

To check the assumptions of the above theorem for our concrete operator, we use the following key proposition.

Proposition 2.9. *Let $S_\gamma = \lambda' S + \gamma I$, $H_\gamma = S_\gamma + T_\mu$ such that $\lambda' > 0, \gamma > 0$ and $H_{\lambda',\mu} = \lambda' A^{*2}A^2 + \mu A^*A + i\lambda A^*(A^* + A)A$ with $i^2 = -1, \lambda' \geqslant 0$ and $\mu > 0$. Then we have*

(1) $\|S_\gamma^{-1}\| \leqslant \frac{1}{\gamma}$ *for every* $\gamma > 0$.
(2) $\|(I + \gamma H_{\lambda',\mu})^{-1}\| \leqslant 1$ *for every* $\gamma \geqslant 0$.
(3) *For all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$\big\|T_\mu S_\gamma^{-1}\big\| \leqslant C_\gamma \quad \text{and} \quad \big\|H_\gamma S_\gamma^{-1}\big\| \leqslant 1 + C_\gamma,$$

where $C_\gamma = \frac{2\varepsilon}{\lambda'} + \frac{C_\varepsilon}{\gamma}$.
(4) *For a sufficiently large γ, there is a constant C_γ, $0 < C_\gamma < 1$, such that*

$$\|T_\mu\phi\| \leqslant C_\gamma\|S_\gamma\phi\| \quad \text{for every } \phi \in D(S).$$

Proof. (1) Let $S = A^{*2}A^2$ acting in Bargmann space E with domain $D(S) = \{\phi \in E;\ S\phi \in E\}$. S is an operator of the form T^*T where T is a closed and densely linear operator, so we deduce from the fundamental theorem of Von Neumann (see Kato [28, p. 275]) that $\|(I + \beta S)^{-1}\| \leqslant 1$ and $\mathrm{Im}(I + \beta S) = E$ for every $\beta > 0$. Now we choose $\beta = \frac{\lambda'}{\gamma}$ to obtain $\|S_\gamma^{-1}\| \leqslant \frac{1}{\gamma}$ for every $\gamma > 0$.

(2) Let $\lambda' \geqslant 0$ and $\mu > 0$. As $D(H_{\lambda',\mu}) \subset D(H_I^{\min})$ and $H_I^{\min}$ is a symmetric operator, then we can write the real part of $\langle (I + \gamma H_{\lambda',\mu})\phi, \phi\rangle$ as

$$\mathrm{Re}\big\langle (I + \gamma H_{\lambda',\mu})\phi, \phi\big\rangle = \|\phi\|^2 + \gamma\lambda' \big\|A^2\phi\big\|^2 + \gamma\mu\|A\phi\|^2, \quad \phi \in D(H).$$

By virtue of the above equality and by using the Cauchy inequality, we get

$$\|\phi\|^2 \leqslant \mathrm{Re}\big\langle (I + \gamma H_{\lambda',\mu})\phi, \phi\big\rangle$$

and

$$\big\|(I + \gamma H_{\lambda',\mu})^{-1}\big\| \leqslant 1 \quad \text{for all } \gamma \geqslant 0.$$

(3) By virtue of inequality (6) of Theorem 1.2, we see that for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\begin{aligned}\big\|T_\mu S_\gamma^{-1}\phi\big\| &\leqslant \varepsilon\big\|SS_\gamma^{-1}\phi\big\| + C_\varepsilon\big\|S_\gamma^{-1}\phi\big\|, \quad \phi \in E\\ &\leqslant \frac{\varepsilon}{\lambda'}\big\|(\lambda' S + \gamma I - \gamma I)S_\gamma^{-1}\phi\big\| + C_\varepsilon\big\|S_\gamma^{-1}\phi\big\|, \quad \phi \in E\\ &\leqslant \frac{\varepsilon}{\lambda'}\big[\|\phi\| + \gamma\big\|S_\gamma^{-1}\phi\big\|\big] + C_\varepsilon\big\|S_\gamma^{-1}\phi\big\|, \quad \phi \in E.\end{aligned}$$

Now, it follows from the property (1) of this proposition, that

$$\big\|T_\mu S_\gamma^{-1}\phi\big\| \leqslant \left(\frac{2\varepsilon}{\lambda'} + \frac{C_\varepsilon}{\gamma}\right)\|\phi\|, \quad \phi \in E,$$

and

$$\big\|T_\mu S_\gamma^{-1}\big\| \leqslant \frac{2\varepsilon}{\lambda'} + \frac{C_\varepsilon}{\gamma}.$$

As $H_\gamma = S_\gamma + T_\mu$, it follows that $H_\gamma S_\gamma^{-1} = I + T_\mu S_\gamma^{-1}$ and we obtain

$$\big\|H_\gamma S_\gamma^{-1}\big\| \leqslant 1 + \frac{2\varepsilon}{\lambda'} + \frac{C_\varepsilon}{\gamma}.$$

(4) By choosing $\varepsilon < \frac{\lambda'}{2}$ and $\gamma > \frac{C_\varepsilon}{1-\frac{2\varepsilon}{\lambda'}}$ in (3), we deduce that $\frac{2\varepsilon}{\lambda'} + \frac{C_\varepsilon}{\gamma} < 1$ and there exists

$$C_\gamma = \frac{2\varepsilon}{\lambda'} + \frac{C_\varepsilon}{\gamma} \quad \text{with } 0 < C_\gamma < 1 \quad \text{and} \quad \|T_\mu\phi\| \leqslant C_\gamma\|S_\gamma\phi\|, \quad \phi \in D(S). \qquad \square$$

Theorem 2.10. *Let $e^{-tH_{\lambda'}}$ and e^{-tT_μ} be the semigroups generated by $H_{\lambda'}$ and T_μ, respectively. Then*

- (i) $e^{-tH_{\lambda'}}$ *converges strongly to* e^{-tT_μ}, $t \geqslant 0$ *as* $\lambda' \to 0$; *the convergence is uniform with respect to t in each finite subinterval of* $[0, \infty[$.
- (ii) *For every* $\xi \in \mathbb{C}$ *with* $\mathrm{Re}\,\xi > 0$ *there is a constant* $C_\xi > 0$ *such that*

$$\big\|\big(H'_\lambda + \xi I\big)^{-1}\big(e^{-tH_{\lambda'}} - e^{-tT_\mu}\big)(T_\mu + xI)^{-1}\big\| \leqslant tC_\xi\sqrt{\lambda'}, \quad t \geqslant 0.$$

(iii) $e^{-tH_{\lambda'}}$ *is given by Trotter product formula* $e^{-tH_{\lambda'}} = \mathrm{Lim}[e^{-t\lambda'/nS}e^{-t/nTm}]^n$ *as* $n \to \infty$ *and we have*

$$\operatorname*{Lim}_{\lambda'\to 0}\Bigl(\operatorname*{Lim}_{n\to\infty}\bigl[e^{-t\lambda'/nS}e^{-t/nT_\mu}\bigr]^n\Bigr) = \operatorname*{Lim}_{n\to\infty}\Bigl(\operatorname*{Lim}_{\lambda'\to 0}\bigl[e^{-t\lambda'/nS}e^{-t/nT_\mu}\bigr]^n\Bigr).$$

(iv) *For* $\lambda' > 0$ *there is a constant* $C > 0$ *such that*

$$\bigl\|\bigl(e^{-(t/n)\lambda'S}e^{-(t/n)T_\mu}\bigr)^n - e^{-tH_{\lambda'}}\bigr\|_1 \leqslant C\frac{\log n}{n}, \quad n = 2, 3, \ldots.$$

Proof. (i) follows immediately from Theorem 2.16 in Kato's book [28, Chapter IX, p. 504].

(ii) is easy to show by applying the above Theorem 2.3 and Theorem 2.14 in Kato's book [28, Chapter IX, p. 503].

(iii) Using Theorem X.51 in the book of Reed–Simon [36, Tome II, p. 245], we deduce the convergence of the usual Trotter formula for our operator $H_{\lambda'}$ and as $e^{-t\lambda'/nS}$ tends to the identity strongly as $\lambda' \to 0$, it follows that

$$e^{-tT_\mu} = \mathrm{Lim}\bigl[e^{-t\lambda'/nS}e^{-t/nT_\mu}\bigr]^n \quad \text{as } \lambda' \to 0.$$

Thus we obtain the equality

$$\operatorname*{Lim}_{\lambda'\to 0}\Bigl(\mathrm{Lim}_{n\to\infty}\bigl[e^{-t\lambda'/nS}e^{-t/nT_\mu}\bigr]^n\Bigr) = \operatorname*{Lim}_{n\to\infty}\Bigl(\operatorname*{Lim}_{\lambda'\to 0}\bigl[e^{-t\lambda'/nS}e^{-t/nT_\mu}\bigr]^n\Bigr).$$

(iv) As $T_\mu^* = \mu A^*A - i\lambda H_I$ with $D(T_\mu^*) = D(T_\mu)$, then it follows from the assertions (1) and (8) of Theorem 1.2 and from (4) of Proposition 2.9 that the assumptions of Cachia–Zagrebnov's theorem recalled above, are checked and so we get the error bound estimate. □

3. A generalized approximation of the evolution operator $e^{-tH_{\lambda'}}$

If $H_{\lambda'}$ is split in the sum of $\lambda' A^{*2}A^2 + \mu A^*A$ and $i\lambda A^*(A^* + A)A$, i.e., $H_{\lambda'} = G + i\lambda H_I$ where $G = \lambda' S + \mu U = \lambda' A^{*2}A^2 + \mu A^*A$ and $H_I = A^*(A + A^*)A$, the results of Cachia–Neidhardt–Zagrebnov [8–10,30–33] are not applicable because the cubic operator H_I is not the infinitesimal generator of a semigroup on Bargmann space. This remark is an argument among others for which the usual Trotter product formula does not hold.

Let $H_I = A^*(A + A^*)A$ with maximal domain $D_{\max}(H_I) = \{\phi \in E$ such that $H_I\phi \in E\}$ and $H_I^{\min} = A^*(A^* + A)A$ with minimal domain $D_{\min}(H_I) = \{\phi \in E$ such that there exist $p_n \in P_0$ and $\psi \in E$ with $p_n \xrightarrow[n\to\infty]{} \phi$ and $H_I p_n \xrightarrow[n\to\infty]{} \psi\}$ where P_0 is the space of polynomials vanishing at the origin; then we can note others subtle properties [13,24] of H_I:

(i) $\sigma(H_I^{\min}) = \mathbb{C}$.
(ii) $(H_I^{\min})^* = H_I = A^*(A^* + A)A$ with domain $D_{\max}(H_I)$.
(iii) $D_{\min}(H_I) \neq D_{\max}(H_I)$.
(iv) $D(A^*A) \not\subset D_{\max}(H_I)$ and $D_{\max}(H_I) \not\subset D(A^*A)$.

These properties indicate that one has to pay attention to the domain on which the Gribov's operator is defined, see for example [34,38] for other concrete operators.

The main problem is to look for an operator θ_ε such that

$$\operatorname{Lim}(\theta_\varepsilon)^n = e^{-tH_{\lambda'}}, \quad \varepsilon = \frac{t}{n} \text{ as } n \to \infty \text{ in strong sense.}$$

A necessary requirement on θ_ε imposed by $\operatorname{Lim}(\theta_\varepsilon)^n = e^{-tH_{\lambda'}}$ as $n \to \infty$ is that

$$\theta_\varepsilon = I - \varepsilon H_{\lambda'} + o(\varepsilon).$$

However, this will not insure in general the existence of the limit. In this section we choose θ_ε as follows:

$$\theta_\varepsilon = e^{-\varepsilon G} - \varepsilon i\lambda A^*\big(A^* e^{-\varepsilon G} + e^{-\varepsilon G} A\big)A.$$

This choice allows us to show that there exists a constant c such that

$$\big\|(\theta_\varepsilon)^n\big\| \leqslant e^{ct}; \quad \varepsilon = \frac{t}{n} \text{ for every } t > 0$$

and to estimate the difference $\theta_\varepsilon\phi - e^{-\varepsilon H_{\lambda'}}\phi$ to order 2 with respect to the parameter ε for all ϕ in $D(G^3)$, i.e.,

$$\frac{1}{\varepsilon}\big(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}}\big)\phi \to 0 \quad \text{as } \varepsilon \to 0 \text{ (see Theorem 3.4 below).}$$

We start with preliminary lemma.

Lemma 3.1. *Let $\phi(z) = \sum_{j=1}^{\infty} \alpha_j \frac{z^j}{\sqrt{j!}}$ and $\psi(z) = \sum_{k=1}^{\infty} \beta_k \frac{z^k}{\sqrt{k!}}$ in the Bargmann space E and*

$$\theta_\varepsilon = e^{-\varepsilon G} - \varepsilon i\lambda A^*\big(A^* e^{-\varepsilon G} + e^{-\varepsilon G} A\big)A \quad \textit{where } G = \lambda' S + \mu U,\ \varepsilon > 0.$$

Then for all $\rho > 0$ we have

(i) $$\big|\big\langle\phi(z), \theta_\varepsilon\psi(z)\big\rangle\big| \leqslant \operatorname{Sup}_j\left[e^{\rho j}\sum_{k=1}^{\infty}|M_{jk}|e^{-\rho k}\right]\|\phi\|\|\psi\|,$$

$$\textit{where } M_{jk} = \left\langle\frac{z^j}{\sqrt{j!}}, \theta_\varepsilon\frac{z^k}{\sqrt{k!}}\right\rangle.$$

(ii) *$\|\theta_\varepsilon\| \leqslant \operatorname{Inf}_\rho \operatorname{Sup}_j U_j(\varepsilon)$ where $U_j(\varepsilon) = e^{-\varepsilon\lambda_j} + \varepsilon|\lambda| j\sqrt{j+1}e^{-\rho-\varepsilon\lambda_{j-1}} + \varepsilon|\lambda|(j-1)\times \sqrt{j}e^{\rho-\varepsilon\lambda_{j-2}}$ and $\lambda_j = \lambda' j(j-1) + \mu j$ are eigenvalues of G associated to $e_j(z) = \frac{z^j}{\sqrt{j!}}$.*

Proof. (i) As $G = \lambda' S + \mu U$, then G acting in Bargmann space is a selfadjoint operator with compact resolvent and generates an analytic semigroup e^{-tG}. The spectral decomposition of G is given by $G\phi = \sum_{k=1}^{\infty} \lambda_k\langle\phi, e_k\rangle e_k$ where $\lambda_k = \lambda' k(k-1) + \mu k$ and $e_k(z) = \frac{z^k}{\sqrt{k!}}$.

Since $\big\langle\frac{z^j}{\sqrt{j!}}, \frac{z^k}{\sqrt{k!}}\big\rangle = \delta_{jk}$ and $e^{-\varepsilon G}\frac{z^k}{\sqrt{k!}} = e^{-\varepsilon\lambda_k}\frac{z^k}{\sqrt{k!}}$, $k = 1, 2, \ldots$ with $\delta_{jk} = 1$ if $j = k$ and 0 if $j \neq k$, it follows that

$$\text{(a)}\quad \left\langle \frac{z^j}{\sqrt{j!}}, A^{*2}e^{-\varepsilon G_A}\frac{z^k}{\sqrt{k!}}\right\rangle = \left\langle \frac{z^j}{\sqrt{j!}}, k\sqrt{k+1}e^{-\varepsilon\lambda_{k-1}}\frac{z^{k+1}}{\sqrt{(k+1)!}}\right\rangle$$
$$= k\sqrt{k+1}e^{-\varepsilon\lambda_{k-1}}\delta_{jk+1};$$

$$\text{(b)}\quad \left\langle \frac{z^j}{\sqrt{j!}}, A^{*}e^{-\varepsilon G}A^2\frac{z^k}{\sqrt{k!}}\right\rangle = \left\langle A^{*2}e^{-\varepsilon G}A\frac{z^j}{\sqrt{j!}}, \frac{z^k}{\sqrt{k!}}\right\rangle$$
$$= j\sqrt{j+1}e^{-\varepsilon\lambda_{j-1}}\delta_{j+1k}.$$

Now, we consider the matrix representation $M_\varepsilon = (M_{jk}) = \langle \frac{z^j}{\sqrt{j!}}, \theta_\varepsilon \frac{z^k}{\sqrt{k!}}\rangle$ of θ_ε in the basis $\{\frac{z^k}{\sqrt{k!}}\}$. The elements of this matrix are given by

$$\begin{aligned}\left\langle \frac{z^j}{\sqrt{j!}}, \theta_\varepsilon\frac{z^k}{\sqrt{k!}}\right\rangle &= \left\langle \frac{z^j}{\sqrt{j!}}, \left(e^{-\varepsilon G} - \varepsilon i\lambda A^*\left(A^*e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A\right)\frac{z^k}{\sqrt{k!}}\right\rangle\\
&= \left\langle \frac{z^j}{\sqrt{j!}}, e^{-\varepsilon G}\frac{z^k}{\sqrt{k!}}\right\rangle - \varepsilon i\lambda\left\langle \frac{z^j}{\sqrt{j!}}, A^{*2}e^{-\varepsilon G}A\frac{z^k}{\sqrt{k!}}\right\rangle\\
&\quad - \varepsilon i\lambda\left\langle \frac{z^j}{\sqrt{j!}}, A^{*}e^{-\varepsilon G}A^2\frac{z^k}{\sqrt{k!}}\right\rangle\\
&= e^{-\varepsilon\lambda_k}\delta_{jk} - \varepsilon i\lambda k\sqrt{k+1}e^{-\varepsilon\lambda_{k-1}}\delta_{jk+1} - \varepsilon i\lambda j\sqrt{j+1}e^{-\varepsilon\lambda_{j-1}}\delta_{j+1k}.\end{aligned}$$

By virtue of

$$(M_{jk}) = \left\langle \frac{z^j}{\sqrt{j!}}, \theta_\varepsilon\frac{z^k}{\sqrt{k!}}\right\rangle, \qquad \phi(z) = \sum_{j=1}^{\infty}\alpha_j\frac{z^j}{\sqrt{j!}} \quad\text{and}\quad \psi(z) = \sum_{k=1}^{\infty}\beta_k\frac{z^k}{\sqrt{k!}},$$

we obtain

$$\langle\phi(z), \theta_\varepsilon\psi(z)\rangle = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\alpha_j M_{jk}\beta_k \quad\text{and}$$

$$\left|\langle\phi(z), \theta_\varepsilon\psi(z)\rangle\right| \leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\alpha_j||M_{jk}||\beta_k|.$$

Therefore for all $\rho > 0$ we have

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\alpha_j||M_{jk}||\beta_k| = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\alpha_j|\sqrt{|M_{jk}|}\sqrt{e^{\rho(j-k)}}\sqrt{|M_{jk}|}\sqrt{e^{\rho(k-j)}}|\beta_k|$$

and by applying the Cauchy inequality, we find

$$\left|\langle\phi(z), \theta_\varepsilon\psi(z)\rangle\right| \leqslant \sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\alpha_j|^2|M_{jk}|e^{\rho(j-k)}}.\sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|M_{jk}|e^{\rho(k-j)}|\beta_k|^2}.$$

Let

$$M_1 = \mathrm{Sup}_j\left[e^{\rho j}\sum_{k=1}^{\infty}|M_{jk}|e^{-\rho k}\right] \quad\text{and}\quad M_2 = \mathrm{Sup}_k\left[e^{\rho k}\sum_{j=1}^{\infty}|M_{jk}|e^{-\rho j}\right];$$

then we infer that

$$\left|\langle\phi(z),\theta_\varepsilon\psi(z)\rangle\right| \leqslant \sqrt{M_1}\sqrt{\sum_{j=1}^{\infty}|\alpha_j|^2}\sqrt{M_2}\sqrt{\sum_{k=1}^{\infty}|\beta_k|^2},$$

i.e.,

$$\left|\langle\phi(z),\theta_\varepsilon\psi(z)\rangle\right| \leqslant \sqrt{M_1}\sqrt{M_2}\|\phi\|\ \|\psi\|.$$

Since (M_{jk}) is symmetric, then $M_1 = M_2$ and for all $\rho > 0$ we have

$$\left|\langle\phi(z),\theta_\varepsilon\psi(z)\rangle\right| \leqslant \mathrm{Sup}_j\left[e^{\rho j}\sum_{k=1}^{\infty}|M_{jk}|e^{-\rho k}\right]\|\phi\|\|\psi\|.$$

(ii) We define the norm of θ_ε by $\|\theta_\varepsilon\| = \mathrm{Sup}\,|\langle\theta_\varepsilon\phi,\psi\rangle|$; $\|\phi\| = \|\psi\| = 1$ and we infer that

$$\|\theta_\varepsilon\| \leqslant \mathrm{Sup}_j\left[e^{\rho j}\sum_{k=1}^{\infty}|M_{jk}|e^{-\rho k}\right].$$

Hence we finally obtain

$$\begin{aligned}&\|\theta_\varepsilon\| \leqslant \mathrm{Inf}_\rho\,\mathrm{Sup}_j\,U_j(\varepsilon), \quad \text{where}\\ &U_j(\varepsilon) = e^{-\varepsilon\lambda_j} + \varepsilon|\lambda|j\sqrt{j+1}e^{-\rho-\varepsilon\lambda_{j-1}} + \varepsilon|\lambda|(j-1)\sqrt{j}e^{\rho-\varepsilon\lambda_{j-2}}.\end{aligned} \qquad \square$$

Now, we have all the ingredients to prove the main lemma of this section:

Lemma 3.2.

(i) *For sufficiently small ε we have*

$$\begin{aligned}&U_j(\varepsilon) \simeq (1+\varepsilon C_j), \quad \textit{where}\\ &C_j = -\lambda_j + |\lambda|j\sqrt{j+1}e^{-\rho} + |\lambda|(j-1)\sqrt{j}e^{\rho} \quad \textit{for all } j.\end{aligned}$$

(ii) *For sufficiently large j and fixed ε we have*

$$U_j(\varepsilon) \simeq \left(1+2\varepsilon|\lambda|j^{3/2}\right)e^{-\varepsilon\lambda' j^2}.$$

(iii) *The maximum of $U_j(\varepsilon)$ is reached in $j_0(\varepsilon)$ such that $j_0(\varepsilon) < \infty$ when $\varepsilon \to 0$.*

Proof. (i) Starting from $U_j(\varepsilon) = e^{-\varepsilon\lambda_j} + \varepsilon|\lambda|j\sqrt{j+1}e^{-\rho-\varepsilon\lambda_{j-1}} + \varepsilon|\lambda|(j-1)\sqrt{j}e^{\rho-\varepsilon\lambda_{j-2}}$, we note that for fixed j and $\varepsilon \to 0$ we have

$$U_j(\varepsilon) \simeq 1 - \varepsilon\lambda_j + \varepsilon|\lambda|j\sqrt{j+1}e^{-\rho}(1-\varepsilon\lambda_{j-1}) + \varepsilon|\lambda|(j-1)\sqrt{j}e^{\rho}(1-\varepsilon\lambda_{j-2}),$$

i.e.,

$$U_j(\varepsilon) \simeq (1+\varepsilon C_j), \quad \text{where } C_j = -\lambda_j + |\lambda|j\sqrt{j+1}e^{-\rho} + |\lambda|(j-1)\sqrt{j}e^{\rho}.$$

(ii) As $\lambda_j = \lambda' j(j-1) + \mu j$, then for sufficiently large j we have

$$\lambda_j \simeq \lambda_{j-1} \simeq \lambda_{j-2} \simeq \lambda' j^2 \quad \text{and} \quad j\sqrt{j+1} \simeq (j-1)\sqrt{j} \simeq j^{3/2}.$$

It follows that

$$U_j(\varepsilon) \simeq \big(1 + 2\varepsilon|\lambda| j^{3/2}\big)e^{-\varepsilon\lambda' j^2}.$$

(iii) Assume, on the contrary, that $j_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$. By using (ii), we obtain

$$\operatorname{Sup}_j U_j(\varepsilon) \simeq \operatorname{Sup}_j \big(1 + 2\varepsilon|\lambda| j^{3/2}\big)e^{-\varepsilon\lambda' j^2} = \big(1 + 2\varepsilon|\lambda| j_0(\varepsilon)^{3/2}\big)e^{-\varepsilon\lambda' j_0(\varepsilon)^2}.$$

In addition, if we assume that $\varepsilon j_0^2(\varepsilon) \to \infty$ as $\varepsilon \to 0$, then we infer that

$$-\varepsilon\lambda' j_0^2(\varepsilon) \to -\infty \quad \text{and} \quad \big(1 + 2\varepsilon|\lambda| j_0(\varepsilon)^{3/2}\big)e^{-\varepsilon\lambda' j_0(\varepsilon)^2} \to 0$$
$$\text{as } \varepsilon \to 0 \text{ because } \lambda' > 0.$$

Therefore, we have the contradiction $\operatorname{Sup}_j U_j(\varepsilon) \to 0$ as $\varepsilon \to 0$.

In particular, $j_0(\varepsilon)$ cannot grow more than $\frac{1}{\sqrt{\varepsilon}}$, so we have $j_0(\varepsilon) = \frac{v(\varepsilon)}{\sqrt{\lambda'\varepsilon}}$ with $v(\varepsilon)$ as bounded function. In this case we have

$$\operatorname{Sup}_j U_j(\varepsilon) \simeq \big(1 + 2\varepsilon|\lambda|(\lambda'\varepsilon)^{-3/4} v^{3/2}(\varepsilon)\big)e^{-v^2}(\varepsilon).$$

Since

$$\begin{aligned}
&\big(1 + 2\varepsilon|\lambda|(\lambda'\varepsilon)^{-3/4} v^{3/2}(\varepsilon)\big) \\
&\quad = \left(1 + 2\left|\frac{\lambda}{\lambda'}\right|(\lambda'\varepsilon)^{1/4} v^{3/2}(\varepsilon)\right) = \left(1 + 2\left|\frac{\lambda}{\lambda'}\right|(\lambda'\varepsilon)^{1/4} v^{-1/2}(\varepsilon) v^2(\varepsilon)\right) \\
&\quad = \left(1 + 2\left|\frac{\lambda}{\lambda'}\right|\frac{1}{\sqrt{j_0(\varepsilon)}} v^2(\varepsilon)\right) \leqslant \operatorname{Exp}\left(2\left|\frac{\lambda}{\lambda'}\right|\frac{1}{\sqrt{j_0(\varepsilon)}} v^2(\varepsilon)\right),
\end{aligned}$$

it follows that

$$\begin{aligned}
\big(1 + 2\varepsilon|\lambda| j_0(\varepsilon)^{3/2}\big)e^{-\varepsilon\lambda' j_0(\varepsilon)^2} &\leqslant \operatorname{Exp}\left(-v^2(\varepsilon) + 2\left|\frac{\lambda}{\lambda'}\right|\frac{1}{\sqrt{j_0(\varepsilon)}} v^2(\varepsilon)\right) \\
&\leqslant \operatorname{Exp}\left(-v^2(\varepsilon)\left[1 - 2\left|\frac{\lambda}{\lambda'}\right|\frac{1}{\sqrt{j_0}(\varepsilon)}\right]\right).
\end{aligned}$$

Now by using $U_j(\varepsilon) \simeq (1 + \varepsilon C_j)$ with $C_j = -\lambda_j + |\lambda| j\sqrt{j+1}e^{-\rho} + |\lambda|(j-1)\sqrt{j}e^{\rho}$ for finite j (see (i) of this lemma) and because $\frac{v^2(\varepsilon)}{\varepsilon}$ is sufficiently large for sufficiently small ε, we get

$$-\frac{v2(\varepsilon)}{\varepsilon} < C_j \quad \text{for sufficiently small } \varepsilon,$$

and

$$\operatorname{Exp}\big(-v^2(\varepsilon)\big) < \operatorname{Exp}(\varepsilon C_j) \simeq 1 + \varepsilon C_j \quad \text{for sufficiently small } \varepsilon.$$

In particular,

$$\operatorname{Exp}\big(-v^2(\varepsilon)\big) < \operatorname{Sup}_j U_j(\varepsilon) \quad \text{for sufficiently small } \varepsilon.$$

This contradicts the assumption that $j_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$. $\square$

Corollary 3.3. *Let $G = \lambda' S + \mu U$ and $\theta_\varepsilon = e^{-\varepsilon G} - \varepsilon i\lambda A^*(A^* e^{-\varepsilon G} + e^{-\varepsilon G} A)A$. Then*

(i) *There is a constant* $c > 0$ *such that* $|\theta_\varepsilon\| \leqslant 1 + c\varepsilon$.
(ii) $\|(\theta_\varepsilon)^n\| \leqslant e^{ct}$; $\varepsilon = \frac{t}{n}$ *for every* $t > 0$.

Proof. (i) is obtained by using Lemmas 3.1 and 3.2.
(ii) is easy to show from (i). $\square$

In the following theorem we establish the connection between θ_ε and $e^{-\varepsilon H_{\lambda'}}$.

Theorem 3.4.

(i) *For* $\varepsilon > 0$, *there exists* $L_\phi > 0$ *such that* $\|(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}})\phi\| \leqslant L_\phi \varepsilon^2$, $\phi \in D(G^3)$.
(ii) $\theta_\varepsilon = I - \varepsilon H_{\lambda'} + o(\varepsilon)$.
(iii) *For* $\varepsilon > 0$, *there exists* $M_\phi > 0$ *such that* $\|([I - i\varepsilon\lambda H_I]e^{-\varepsilon G} - \theta_\varepsilon)\phi\| \leqslant M_\phi \varepsilon^2$, $\phi \in D(G^3)$.

Proof. (i) By virtue of semigroup properties of $e^{-\varepsilon H_{\lambda'}}$, we have for $\varepsilon > 0$,

$$\left(e^{-\varepsilon H_{\lambda'}} - I\right)\phi = -\int_0^\varepsilon e^{-sH_{\lambda'}} H_{\lambda'}\phi\, ds \quad \text{for } \phi \in D(H_{\lambda'}) \quad \text{and}$$

$$\left\|\left(e^{-\varepsilon H_{\lambda'}} - I\right)\phi\right\| \leqslant \varepsilon \|H_{\lambda'}\phi\|$$

for $\phi \in D(H_{\lambda'})$. Now since $H_{\lambda'}$ is boundedly invertible, we have

$$\begin{aligned}\left[e^{-\varepsilon H_{\lambda'}} - (I - \varepsilon H_{\lambda'})\right]H_{\lambda'}^{-2}\psi &= \left(e^{-\varepsilon H_{\lambda'}} - I\right)H_{\lambda'}^{-2}\psi + \varepsilon H_{\lambda'}^{-1}\psi \\ &= -\int_0^\varepsilon \left(e^{-sH_{\lambda'}} - I\right)H_{\lambda'}^{-1}\psi\, ds.\end{aligned}$$

Therefore,

$$\left\|\left[e^{-\varepsilon H_{\lambda'}} - (I - \varepsilon H_{\lambda'})\right]H_{\lambda'}^{-2}\psi\right\| \leqslant \int_0^\varepsilon s\|\psi\|\, ds \leqslant \frac{\varepsilon^2}{2}\|\psi\| \quad \text{for } \psi \in E,$$

and finally,

$$\left\|\left[e^{-\varepsilon H_{\lambda'}} - (I - \varepsilon H_{\lambda'})\right]\phi\right\| \leqslant \frac{\varepsilon^2}{2}\left\|H_{\lambda'}^2\phi\right\| \quad \text{for } \phi \in D\left(H_{\lambda'}^2\right).$$

On the same way, as G is boundedly invertible since $\lambda' > 0$ and $\mu > 0$, we have

$$\left\|\left[e^{-\varepsilon G} - (I - \varepsilon G)\right]G^{-2}\psi\right\| \leqslant \frac{\varepsilon^2}{2}\|\psi\| \quad \text{for } \psi \in E \quad \text{and}$$

$$\left\|\left[e^{-\varepsilon G} - (I - \varepsilon G)\right]\phi\right\| \leqslant \frac{\varepsilon^2}{2}\left\|G^2\phi\right\| \quad \text{for } \phi \in D\left(G^2\right).$$

Now we can write $(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}})\phi$ in the following form:

$$\begin{aligned}\left(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}}\right)\phi = \big[&e^{-\varepsilon G} - \varepsilon i\lambda A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A \\ &- (I - \varepsilon H_{\lambda'}) + (I - \varepsilon H_{\lambda'}) - e^{-\varepsilon H_{\lambda'}}\big]\phi.\end{aligned}$$

Then

$$\begin{aligned}\left\|\left(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}}\right)\phi\right\| \leqslant{}& \left\|\left[e^{-\varepsilon G} - (I-\varepsilon G)\right]\phi\right\| + \varepsilon|\lambda| \left\|\left[A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A\right.\right. \\ &\left.\left. - A^*(A+A^*)A\right]\phi\right\| + \left\|\left[(I-\varepsilon H_{\lambda'}) - e^{-\varepsilon H_{\lambda'}}\right]\phi\right\|,\end{aligned}$$

which can be written as

$$\begin{aligned}\left\|\left(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}}\right)\phi\right\| \leqslant{}& \frac{\varepsilon^2}{2}\left\|G^2\phi\right\| + \varepsilon|\lambda| \left\|\left[A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A\right.\right. \\ &\left.\left. - A^*\left(A+A^*\right)A\right]\phi\right\| + \frac{\varepsilon^2}{2}\left\|H_{\lambda'}^2\phi\right\|.\end{aligned}$$

To conclude, it remains to give an estimation of the term

$$\left\|\left[A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A - A^*\left(A+A^*\right)A\right]\phi\right\|.$$

To this aim, we first begin to notice that for $\lambda' > 0$ and $\mu > 0$ we have

$$\langle S\phi, U\phi\rangle \geqslant 0 \quad \text{and} \quad \|S\phi\| \leqslant \frac{1}{\lambda'}\left\|(\lambda' S + \mu U)\phi\right\| \quad \text{since}$$
$$\langle S\phi, U\phi\rangle = \left\|A^*A^2\phi\right\|^2 + \left\|A^2\phi\right\|^2.$$

Therefore $\|S\phi\| \leqslant \frac{1}{\lambda'}\|G\phi\|$ for every $\phi \in D(G)$.

From the property (1) of Lemma 2.6 combined with the property (3) of Remark 1.1, we deduce that for all $\delta > 0$ there exists $C_\delta > 0$ such that $\|A^{*2}\phi\| \leqslant \frac{\delta}{\lambda'}\|G\phi\| + C_\delta\|\phi\|$ and

$$\left\|A^*\phi\right\| \leqslant \frac{\delta}{\lambda'\sqrt{2}}\|G\phi\| + \frac{C_\delta}{\sqrt{2}}\|\phi\| \quad \text{for every } \phi \in D(G).$$

Now as

$$\begin{aligned}&\left\|\left[A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A - A^*\left(A+A^*\right)A\right]\phi\right\| \\ &\quad = \left\|A^{*2}\left(e^{-\varepsilon G} - I\right)A\phi + A^*\left(e^{-\varepsilon G} - I\right)A^2\phi\right\| \\ &\quad \leqslant \left\|A^{*2}\left(e^{-\varepsilon G} - I\right)A\phi\right\| + \left\|A^*\left(e^{-\varepsilon G} - I\right)A^2\phi\right\|,\end{aligned}$$

we obtain

$$\begin{aligned}&\left\|\left[A^*\left(A^* e^{-\varepsilon G} + e^{-\varepsilon G}A\right)A - A^*\left(A+A^*\right)A\right]\phi\right\| \\ &\quad \leqslant \frac{\delta}{\lambda'}\left\|G\left(e^{-\varepsilon G} - I\right)A\phi\right\| + C_\delta\left\|\left(e^{-\varepsilon G} - I\right)A\phi\right\| + \frac{\delta}{\lambda'\sqrt{2}}\left\|G\left(e^{-\varepsilon G} - I\right)A^2\phi\right\| \\ &\qquad + \frac{C_\delta}{\sqrt{2}}\left\|\left(e^{-\varepsilon G} - I\right)A^2\phi\right\| \\ &\quad \leqslant \varepsilon\left(\frac{\delta}{\lambda'}\|GA\phi\| + C_\delta\|A\phi\| + \frac{\delta}{\lambda'\sqrt{2}}\left\|GA^2\phi\right\| + \frac{C_\delta}{\sqrt{2}}\left\|A^2\phi\right\|\right) \\ &\quad \text{for every } \phi \in D\left(G^3\right).\end{aligned}$$

If we put

$$\begin{aligned}L_\phi ={}& |\lambda|\left(\frac{\delta}{\lambda'}\|GA\phi\| + C_\delta\|A\phi\| + \frac{\delta}{\lambda'\sqrt{2}}\left\|GA^2\phi\right\| + \frac{C_\delta}{\sqrt{2}}\left\|A^2\phi\right\|\right) \\ &+ \frac{1}{2}\left(\left\|G^2\phi\right\| + \left\|H_{\lambda'}^2\phi\right\|\right),\end{aligned}$$

we obtain

$$\left\|\left(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}}\right)\phi\right\| \leqslant L_\phi \varepsilon^2 \quad \text{for all } \phi \in D\left(G^3\right).$$

(ii) is a consequence of (i) and the density of $D(G^3)$.

(iii) is obtained by using the properties of Lemma 2.6. □

Remark 3.5. In further investigation [26], we use the results of this section and the property (9) of Theorem 1.2 to show that

(i) $\frac{1}{\varepsilon}(\theta_\varepsilon - e^{-\varepsilon H_{\lambda'}})\phi \to 0$ as $\varepsilon \to 0$ for every ϕ in the linear span of principal vectors of $H_{\lambda'}$ for $\lambda' > 0$.

(ii) The solution of the Cauchy equation associated to $H_{\lambda'}$ for $\lambda' > 0$ is represented by a "modified Feynman path integral."
Specifically, one of the quantities which have physical interpretation for high-energy scattering process is $\langle 0|e^{i\alpha A} e^{-tH_{\lambda'}} e^{-i\beta A^*}|0\rangle$ where $(\alpha, \beta) \in \mathbb{C}^2$ and $|0\rangle$ is the vacuum state defined by $A|0\rangle = 0$, then we have

$$\begin{aligned}&\left\langle 0\middle|e^{-i\alpha A} e^{-tH_{\lambda'}} e^{-i\beta A^*}\middle|0\right\rangle \\ &\quad = \operatorname*{Lim}_{n\to\infty} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \prod_{k=1}^{n} \frac{dz_k\, d\bar{z}_k}{2i\pi} N_\varepsilon(z_{k+1}, \bar{z}_k) e^{(z_{k+1}-z_k)\bar{z}_{k_e} - i\beta z_1},\end{aligned}$$

where

$$N_\varepsilon(z, \bar{\xi}) = \frac{\langle z|\theta_\varepsilon|\xi\rangle}{\langle z|\xi\rangle} \quad \text{and} \quad z_{n+1} = -i\alpha.$$

Here the coherent state vector $|\xi\rangle$ is defined by

$$|\xi\rangle = e^{-(1/2)|\xi|^2} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |n\rangle,$$

where $\{|n\rangle;\ n = 0, 1, 2, \ldots\}$ are the eigenvectors of the positive semi-definite number operator A^*A and

$$\langle z|\xi\rangle = e^{-(1/2)|z-\xi|^2 + i\,\mathrm{Im}(\bar{z}\xi)}.$$

(iii) Notice that the case $\lambda' = 0$ is entirely different, in fact θ_ε is not uniformly bounded in this case. Nevertheless, with some modifications, the calculations of this section allow us to give a "generalized Trotter product formula" for

$$T_\mu = \mu U + i\lambda H_I, \quad \text{where } U = A^*A \text{ and } H_I = A^*\left(A + A^*\right)A.$$

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